

**Equations of Associativity
in Two-Dimensional Topological Field Theory
as Integrable Hamiltonian Nondiagonalizable
Systems of Hydrodynamic Type †**

O.I.Mokhov

*Department of Geometry and Topology
Steklov Mathematical Institute
ul. Vavilova, 42
Moscow, GSP-1, 117966, Russia
e-mail: mokhov@mian.su*

and

E.V.Ferapontov

*Institute of Mathematical Modelling
Miusskaya, 4
Moscow, 125047, Russia
e-mail: fer#9@imamod.msk.su*

§ 1. Introduction

Let us consider a function of n independent variables $F(t^1, \dots, t^n)$ satisfying the following two conditions:

1. *The matrix*

$$\eta_{\alpha\beta} = \frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} \quad (\alpha, \beta = 1, \dots, n)$$

is constant and nondegenerate.

Note that the matrix $\eta_{\alpha\beta}$ completely determines dependence of the function F on the fixed variable t^1 .

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2. For all $t = (t^1, \dots, t^n)$ the functions

$$c_{\beta\gamma}^\alpha(t) = \eta^{\alpha\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^\beta \partial t^\gamma} \quad (\text{here } \eta^{\alpha\mu} \eta_{\mu\beta} = \delta_\beta^\alpha)$$

are structural constants of an associative algebra $A(t)$ in n -dimensional space with a basis e_1, \dots, e_n and the multiplication

$$e_\beta \circ e_\gamma = c_{\beta\gamma}^\alpha(t) e_\alpha.$$

The conditions 1 and 2 impose a complicated overdetermined system of nonlinear partial differential equations of the third order on the function F . This system is known in two-dimensional topological field theory as the equations of associativity or the Witten-Dijkgraaf-H.Verlinde-E.Verlinde (WDVV) system ([10–13], all necessary physical motivations and theory of integrability for the equations of associativity can be found in the survey of B.Dubrovin [1]).

For $n = 3$ two essentially different types of dependence of the function F on the fixed variable t^1 were considered by Dubrovin:

$$F = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2}t^1(t^2)^2 + f(t^2, t^3)$$

and

$$F = \frac{1}{6}(t^1)^3 + t^1 t^2 t^3 + f(t^2, t^3).$$

For these cases the equations of associativity reduce to the following two nonlinear equations of the third order for a function $f = f(x, t)$ of two independent variables:

$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt} \tag{1.1}$$

and

$$f_{xxx} f_{ttt} - f_{xxt} f_{xtt} = 1 \tag{1.2}$$

correspondingly.

Following [2,3], introduce new variables

$$a = f_{xxx}, \quad b = f_{xxt}, \quad c = f_{xtt}.$$

As it was shown in the papers [2,3], in new variables the equations (1.1) and (1.2) has the form of 3×3 systems of hydrodynamic type:

$$\begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = (b^2 - ac)_x \end{cases} \quad (1.3)$$

and

$$\begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = ((1 + bc)/a)_x \end{cases} \quad (1.4)$$

correspondingly.

We recall that systems of hydrodynamic type are by definition systems of quasilinear equations of the form

$$u_t^i = v_j^i(u)u_x^j.$$

The main advantage of representation of the equations of associativity in the form (1.3), (1.4) is the presence of efficient and elaborate theory of integrability of systems of hydrodynamic type — see, for example, the surveys of Tsarev [4], Dubrovin and Novikov [5], and also the papers [6–9] devoted to systems of hydrodynamic type, which do not possess Riemann invariants.

Everywhere in this paper we consider only strictly hyperbolic systems, i.e., the eigenvalues of the matrix v_j^i are real and distinct. Thus, both the systems under consideration (1.3) and (1.4) are strictly hyperbolic.

In § 2 a Hamiltonian property of the system (1.3) is established. For this system a local nondegenerate Hamiltonian structure of hydrodynamic type (a Poisson bracket of Dubrovin-Novikov type [5]) is found. In contrast to (1.3), the integrable system of hydrodynamic type (1.4) possesses only nonlocal Hamiltonian structures of hydrodynamic type (see [14–16]). Investigation, which was made in [2,3], showed that both these systems (1.3) and (1.4) are nondiagonalizable (i.e., do not possess Riemann invariants).

In § 3 we consider general theory of integrability of nondiagonalizable Hamiltonian 3×3 systems of hydrodynamic type following [6–9]. It turns out, that any such a system can be reduced to the integrable three wave system by some standard chain of transformations. We shall demonstrate this procedure for the system (1.3). Correspondingly, it is shown that

any solution of the integrable three wave system generates solutions of the equation of associativity (1.1).

In § 4 an explicit Bäcklund type transformation connecting solutions of the systems (1.3) and (1.4) is found.

§ 2. Hamiltonian representation of the system (1.3)

As it was noticed by Dubrovin [1], the equation (1.1) is connected with a spectral problem, which has the following form in the variables a, b, c :

$$\begin{aligned} \Psi_x &= zA\Psi = z \begin{pmatrix} 0 & 1 & 0 \\ b & a & 1 \\ c & b & 0 \end{pmatrix} \Psi, \\ \Psi_t &= zB\Psi = z \begin{pmatrix} 0 & 0 & 1 \\ c & b & 0 \\ b^2 - ac & c & 0 \end{pmatrix} \Psi. \end{aligned} \quad (2.1)$$

Compatibility conditions for the spectral problem (2.1) are equivalent to the following two relations between the matrices A, B :

$$\begin{cases} A_t = B_x, \\ [A, B] = 0, \end{cases} \quad (2.2)$$

which are satisfied identically by virtue of the equations (1.3) (here $[\ , \]$ denotes the commutator).

Lemma 1. *The eigenvalues of the matrix A are densities of conservation laws of the system (1.3).*

Proof. So the matrices A and B commute and have simple spectrum, they can be diagonalized simultaneously

$$A = PUP^{-1}, \quad B = PVP^{-1}.$$

Here $U = \text{diag } (u^1, u^2, u^3)$, $V = \text{diag } (v^1, v^2, v^3)$. Substitution in the equation (2.2) gives

$$[P^{-1}P_t, U] + U_t = [P^{-1}P_x, V] + V_x.$$

It remains to note that the matrices $[P^{-1}P_t, U]$ and $[P^{-1}P_x, V]$ are off-diagonal. Hence,

$$U_t = V_x.$$

The Lemma 1 is proved.

Thus, besides three evident conservation laws with the densities a, b, c the system (1.3) has also three conservation laws with the densities u^1, u^2, u^3 , which are roots of the characteristic equation

$$\det(\lambda E - A) = \lambda^3 - a\lambda^2 - 2b\lambda - c = 0.$$

By virtue of the obvious linear relation $a = u^1 + u^2 + u^3$ among them only five conservation laws with the densities u^1, u^2, u^3, b, c are linearly independent. It is easy to show that the system (1.3) has no other conservation laws of hydrodynamic type, i.e., with densities of the form $h(a, b, c)$.

Let us go over in the equations (1.3) from the variables a, b, c to new field variables u^1, u^2, u^3 , connected with a, b, c by the Viète formulas

$$a = u^1 + u^2 + u^3, \quad b = -\frac{1}{2}(u^1u^2 + u^2u^3 + u^3u^1), \quad c = u^1u^2u^3.$$

For shortening of calculations we note that the matrices A and B are connected by the relation

$$B = A^2 - aA - bE.$$

Hence, the same relation is valid for the corresponding diagonal matrices U and V :

$$V = U^2 - aU - bE.$$

Substituting the expressions for a and b and using the equation (2.2), we obtain the following representation for the system (1.3)

$$U_t = (U^2 - (u^1 + u^2 + u^3)U + \frac{1}{2}(u^1u^2 + u^2u^3 + u^3u^1)E)_x$$

or, in components,

$$\begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}_t = \frac{1}{2} \begin{pmatrix} u^2u^3 - u^1u^2 - u^1u^3 \\ u^1u^3 - u^2u^1 - u^2u^3 \\ u^1u^2 - u^3u^1 - u^3u^2 \end{pmatrix}_x = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \frac{d}{dx} \begin{pmatrix} \partial h / \partial u^1 \\ \partial h / \partial u^2 \\ \partial h / \partial u^3 \end{pmatrix}, \quad (2.3)$$

where $h = c = u^1 u^2 u^3$. Hence, the system under consideration is Hamiltonian with the Hamiltonian operator

$$M = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \frac{d}{dx} \quad (2.4)$$

and the Hamiltonian functional $H = \int c dx = \int u^1 u^2 u^3 dx$. Density of momentum and annihilators of the corresponding Poisson bracket has the following form:

$$2b = -u^1 u^2 - u^2 u^3 - u^3 u^1 \quad (\text{density of momentum}),$$

$$u^1, u^2, u^3 \quad (\text{annihilators}).$$

In the initial variables a, b, c the Hamiltonian operator (2.4) is expressed as

$$M = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2}a & b \\ \frac{1}{2}a & b & \frac{3}{2}c \\ b & \frac{3}{2}c & 2(b^2 - ac) \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & \frac{1}{2}a_x & b_x \\ 0 & \frac{1}{2}b_x & c_x \\ 0 & \frac{1}{2}c_x & (b^2 - ac)_x \end{pmatrix}.$$

We recall that local nondegenerate Hamiltonian operators of hydrodynamic type were introduced and studied by Dubrovin and Novikov (see [5]). It was proved that an operator of the form

$$P^{ij} = g^{ij}(u) \frac{d}{dx} + b_k^{ij}(u) u_x^k, \quad \det[g^{ij}(u)] \neq 0,$$

is Hamiltonian if and only if

- (1) $g^{ij}(u)$ is a metric of zero Riemannian curvature (i.e., simply a flat metric);
- (2) $b_k^{ij}(u) = -g^{is}(u) \Gamma_{sk}^j(u)$, where $\Gamma_{sk}^j(u)$ are the coefficients of the differential geometric connection generated by the metric g^{ij} , i.e., the only symmetric connection compatible with the metric (the Levi-Civita connection).

Correspondingly, Hamiltonian systems of hydrodynamic type, which were considered by Dubrovin and Novikov [5], have the form

$$u_t^i = P^{ij} \frac{\partial H}{\partial u^j},$$

where $H = \int h(u)dx$ is a functional of hydrodynamic type. Nonlocal generalizations of the Hamiltonian theory of systems of hydrodynamic type were discovered in [13] (see also [14-15]). Efficient theory of integrability of diagonalizable Hamiltonian systems of hydrodynamic type, i.e., in other words, Hamiltonian systems of hydrodynamic type, which can be reduced to Riemann invariants:

$$R_t^i = v^i(R)R_x^i,$$

was built by Tsarev [4]. All such systems have infinite number of conservation laws and commuting flows of hydrodynamic type and can be integrated by generalized hodograph method. However, it was shown in the papers [2,3] that the system (1.3) do not possess Riemann invariants. This explains, in particular, the fact that the system (1.3) has only finite number of hydrodynamic type integrals.

General theory of integrability of nondiagonalizable (i.e., not possessing Riemann invariants) Hamiltonian systems of hydrodynamic type started to develop in [6-9]. For three-component systems there were obtained final results.

Theorem 1 [7,8]. *Nondiagonalizable Hamiltonian (with nondegenerate Poisson bracket of hydrodynamic type) 3×3 system of hydrodynamic type is integrable if and only if it is weakly nonlinear.*

We recall that a system of hydrodynamic type

$$u_t^i = v_j^i(u)u_x^j, \quad i, j = 1, \dots, n, \quad (2.5)$$

is called weakly nonlinear if for eigenvalues $\lambda^i(u)$ of matrix $v_j^i(u)$ the following relations

$$L_{\vec{X}^i}(\lambda^i) = 0,$$

where $L_{\vec{X}^i}$ is the Lie derivative along eigenvector \vec{X}^i corresponding to eigenvalue λ^i , are satisfied for any $i = 1, \dots, n$.

There exists a simple and efficient criterion of weak nonlinearity, which does not appeal to eigenvalues and eigenvectors.

Proposition [7]. *A system of hydrodynamic type (2.5) is weakly nonlinear if and only if*

$$(\text{grad } f_1)v^{n-1} + (\text{grad } f_2)v^{n-2} + \dots + (\text{grad } f_n)E = 0,$$

where f_i are coefficients of characteristic polynomial

$$\det(\lambda\delta_j^i - v_j^i(u)) = \lambda^n + f_1(u)\lambda^{n-1} + f_2(u)\lambda^{n-2} + \dots + f_n(u),$$

and v^n denotes n -th power of matrix v_j^i .

As it was shown in [2,3], the systems (1.3) and (1.4) are weakly non-linear.

§ 3 Integrable Hamiltonian 3×3 systems of hydrodynamic type, which do not possess Riemann invariants

Consider a system of hydrodynamic type

$$u_t^i = v_j^i(u)u_x^j. \quad (3.1)$$

Let $\lambda^i(u)$ be eigenvalues of matrix v_j^i , i.e., roots of characteristic equation $\det(v_j^i(u) - \lambda\delta_j^i) = 0$ (we assume that the system under consideration is strictly hyperbolic, i.e., all the roots of the characteristic equation are real and distinct). Denote by $\vec{l}^i(u) = (l_1^i, \dots, l_n^i)$ the left eigenvector of the matrix v_j^i , which corresponds to eigenvalue λ^i , i.e., $l_k^i v_j^k = \lambda^i l_j^i$. Introduce 1-forms $\omega^i = l_k^i du^k$ ($i = 1, \dots, n$). We emphasize that 1-forms ω^i are defined up to normalization $\omega^i \mapsto p^i \omega^i$, $p^i \neq 0$. It is easy to verify that the equations (3.1) can be rewritten in the form of system of exterior equations

$$\omega^i \wedge (dx + \lambda^i dt) = 0, \quad i = 1, \dots, n. \quad (3.2)$$

For the being investigated system (2.3) eigenvalues λ^i and corresponding them left eigenvectors \vec{l}^i have the form:

$$\lambda^1 = -u^1, \quad \vec{l}^1 = (u^2 - u^3, u^1 - u^3, u^2 - u^1),$$

$$\lambda^2 = -u^2, \quad \vec{l}^2 = (u^2 - u^3, u^1 - u^3, u^1 - u^2),$$

$$\lambda^3 = -u^3, \quad \vec{l}^3 = (u^2 - u^3, u^3 - u^1, u^2 - u^1).$$

Hence, the equations (2.3) can be expressed as

$$\omega^i \wedge (dx - u^i dt) = 0, \quad i = 1, 2, 3,$$

where

$$\begin{aligned}\omega^1 &= (u^2 - u^3)du^1 + (u^1 - u^3)du^2 + (u^2 - u^1)du^3, \\ \omega^2 &= (u^2 - u^3)du^1 + (u^1 - u^3)du^2 + (u^1 - u^2)du^3, \\ \omega^3 &= (u^2 - u^3)du^1 + (u^3 - u^1)du^2 + (u^2 - u^1)du^3.\end{aligned}\quad (3.3)$$

Let $B(u)dx + A(u)dt$ and $N(u)dx + M(u)dt$ be two hydrodynamic type integrals of a system of hydrodynamic type (3.1), i.e., the differential 1-forms are closed along solutions of the system. Go over from the variables x, t to new independent variables \tilde{x}, \tilde{t} by means of the following formulas

$$\begin{aligned}d\tilde{x} &= Bdx + Adt, \\ d\tilde{t} &= Ndx + Mdt.\end{aligned}\quad (3.4)$$

Then the system (3.1) transforms to the form

$$u_{\tilde{t}}^i = \tilde{v}_j^i(u)u_{\tilde{x}}^j,$$

where the matrix \tilde{v} is related with v by the formula

$$\tilde{v} = (Bv - AE)(ME - Nv)^{-1}.$$

Using the language of exterior equations we can rewrite the transformed system in the following form

$$\omega^i \wedge (d\tilde{x} + \tilde{\lambda}^i d\tilde{t}) = 0, \quad i = 1, \dots, n, \quad (3.5)$$

where

$$\tilde{\lambda}^i = \frac{\lambda^i B - A}{M - \lambda^i N}. \quad (3.6)$$

Hence, the 1-forms ω^i do not change after the transformations of the form (3.4) while the eigenvalues λ^i transform in according with the formula (3.6).

Theorem 2 [7,8]. *If a 3×3 system of hydrodynamic type (3.1) is weakly nonlinear and Hamiltonian (with nondegenerate Poisson bracket of hydrodynamic type) then there exists a pair of integrals (3.4) of this system such that the corresponding transformed system has constant eigenvalues $\tilde{\lambda}^i$, which can be considered equal to 1, -1, 0 without loss of generality.*

For the system (2.3) the transformation (3.4), the existence of which is established by Theorem 2, has the following form:

$$d\tilde{x} = Bdx + Adt = (u^1 - u^2)dx + u^3(u^2 - u^1)dt,$$

$$d\tilde{t} = Ndx + Mdt = (2u^3 - u^1 - u^2)dx + (2u^1u^2 - u^1u^3 - u^2u^3)dt. \quad (3.7)$$

According to the formula (3.6) the transformed eigenvalues will be equal to 1, -1, 0, correspondingly. Hence, in new independent variables \tilde{x}, \tilde{t} the system (2.3) can be rewritten in the form

$$\omega^1 \wedge (d\tilde{x} + d\tilde{t}) = 0, \quad \omega^2 \wedge (d\tilde{x} - d\tilde{t}) = 0, \quad \omega^3 \wedge d\tilde{x} = 0.$$

Theorem 3 [7,8]. *If a 3×3 system of hydrodynamic type (3.1) is nondiagonalizable, weakly nonlinear and Hamiltonian (with nondegenerate Poisson bracket of hydrodynamic type) then the corresponding 1-forms $\omega^1, \omega^2, \omega^3$ can be normalized such that they will satisfy either the structural equations of the group $SO(3)$:*

$$d\omega^1 = \omega^2 \wedge \omega^3, \quad d\omega^2 = \omega^3 \wedge \omega^1, \quad d\omega^3 = \omega^1 \wedge \omega^2, \quad (3.8)$$

if the signature of the metric, determining the Poisson bracket of hydrodynamic type, is Euclidean, or the structural equations of the group $SO(2, 1)$:

$$d\omega^1 = \omega^2 \wedge \omega^3, \quad d\omega^2 = \omega^3 \wedge \omega^1, \quad d\omega^3 = -\omega^1 \wedge \omega^2 \quad (3.9)$$

(in the case of Lorentzian signature of the metric).

For the system (2.3) the signature of the metric of the Poisson bracket (2.4) is Lorentzian. Hence, the forms (3.3) can be normalized such that they will satisfy the structural equations (3.9). Desired normalization has the form (we will not introduce a new notation for the normalized 1-forms ω^i):

$$\begin{aligned} \omega^1 &= \frac{(u^2 - u^3)du^1 + (u^1 - u^3)du^2 + (u^2 - u^1)du^3}{2(u^2 - u^3)\sqrt{(u^2 - u^1)(u^3 - u^1)}}, \\ \omega^2 &= \frac{(u^2 - u^3)du^1 + (u^1 - u^3)du^2 + (u^1 - u^2)du^3}{2(u^3 - u^1)\sqrt{(u^2 - u^1)(u^2 - u^3)}}, \\ \omega^3 &= \frac{(u^2 - u^3)du^1 + (u^3 - u^1)du^2 + (u^2 - u^1)du^3}{2(u^2 - u^1)\sqrt{(u^3 - u^1)(u^2 - u^3)}} \end{aligned} \quad (3.10)$$

(for definiteness we here consider $u^1 < u^3 < u^2$).

One can verify by direct check that the 1-forms (3.10) satisfy to the structural equations (3.9).

Thus, according to theorems 2 and 3 any nondiagonalizable weakly nonlinear Hamiltonian (with nondegenerate Poisson bracket of hydrodynamic type) 3×3 system of hydrodynamic type can be reduced to the canonical form

$$\omega^1 \wedge (d\tilde{x} + d\tilde{t}) = 0, \quad \omega^2 \wedge (d\tilde{x} - d\tilde{t}) = 0, \quad \omega^3 \wedge d\tilde{x} = 0 \quad (3.11)$$

by suitable transformation (3.4). Moreover, one can consider that for the forms ω^i the structural equations (3.8) or (3.9) are satisfied (note that the transformations of the form (3.4) do not change the structural equations). Introducing in the equations (3.11) the variables p^1, p^2, p^3 in according to the formulas (see [6,7])

$$\omega^1 = p^1(d\tilde{x} + d\tilde{t}), \quad \omega^2 = p^2(d\tilde{x} - d\tilde{t}), \quad \omega^3 = p^3 d\tilde{x}. \quad (3.12)$$

and substituting (3.12) in the structural equations (for definiteness in (3.9)) we obtain the integrable system of three waves

$$\begin{aligned} p_{\tilde{t}}^1 - p_{\tilde{x}}^1 &= -p^2 p^3, \\ p_{\tilde{t}}^2 + p_{\tilde{x}}^2 &= -p^1 p^3, \\ p_{\tilde{t}}^3 &= -2p^1 p^2. \end{aligned} \quad (3.13)$$

Remark. If use explicit coordinate representation of the 1-forms $\omega^i = l_k^i(u)du^k$, then for p^i we obtain expressions of the form $p^i = l_k^i(u)u_{\tilde{x}}^k$. Hence, the change of variables from u^i to p^i is a differential substitution of the first order.

Summing up the described construction for the considered system we can present the transition from the equations (2.3) to the system of three waves (3.13) as two the following steps:

1. The change of variables from x, t to new independent variables \tilde{x}, \tilde{t} in according to the formula (3.7).
2. The change of the field variables from u^1, u^2, u^3 to p^1, p^2, p^3 in according to the following formulas (compare with (3.10)):

$$\begin{aligned}
p^1 &= \frac{(u^2 - u^3)u_{\tilde{x}}^1 + (u^1 - u^3)u_{\tilde{x}}^2 + (u^2 - u^1)u_{\tilde{x}}^3}{2(u^2 - u^3)\sqrt{(u^2 - u^1)(u^3 - u^1)}}, \\
p^2 &= \frac{(u^2 - u^3)u_{\tilde{x}}^1 + (u^1 - u^3)u_{\tilde{x}}^2 + (u^1 - u^2)u_{\tilde{x}}^3}{2(u^3 - u^1)\sqrt{(u^2 - u^1)(u^2 - u^3)}}, \\
p^3 &= \frac{(u^2 - u^3)u_{\tilde{x}}^1 + (u^3 - u^1)u_{\tilde{x}}^2 + (u^2 - u^1)u_{\tilde{x}}^3}{2(u^2 - u^1)\sqrt{(u^3 - u^1)(u^2 - u^3)}}. \tag{3.14}
\end{aligned}$$

Thus, any solution of the integrable three wave system (3.13) generates solutions of the equations of associativity (1.1).

§ 4. Relation between the systems (1.3) and (1.4)

The spectral problem corresponding to the system (1.4) has the form

$$\begin{aligned}
\Psi_x &= zA\Psi = z\begin{pmatrix} 0 & 1 & 0 \\ 0 & b & a \\ 1 & c & b \end{pmatrix}\Psi, \\
\Psi_t &= zB\Psi = z\begin{pmatrix} 0 & 0 & 1 \\ 1 & c & b \\ 0 & (1+bc)/a & c \end{pmatrix}\Psi. \tag{4.1}
\end{aligned}$$

It is easy to verify that the matrix B is related with A by the formula

$$B = \frac{1}{a}(A^2 - bA).$$

Compatibility condition for the spectral problem (4.1)

$$A_t = B_x,$$

rewritten in terms of eigenvalues of matrices A and B (see Lemma 1, § 2), has the form:

$$w_t^i = \left(\frac{1}{a}((w^i)^2 - w^i b) \right)_x, \tag{4.2}$$

where w^i are eigenvalues of the matrix A , i.e., the roots of the characteristic equation

$$\det(\lambda E - A) = \lambda^3 - 2b\lambda^2 + (b^2 - ac)\lambda - a = 0.$$

Expressing a and b by the Viète formula

$$b = \frac{1}{2}(w^1 + w^2 + w^3), \quad a = w^1 w^2 w^3$$

and substituting the expressions in (4.2) we obtain the explicit representation of the equations (1.4) in the coordinates w^i :

$$\begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix}_t = \frac{1}{2} \begin{pmatrix} (w^1 - w^2 - w^3)/w^2 w^3 \\ (w^2 - w^1 - w^3)/w^1 w^3 \\ (w^3 - w^1 - w^2)/w^1 w^2 \end{pmatrix}_x. \quad (4.3)$$

Note that the integrable systems of hydrodynamic type (1.4) and (4.3) do not possess local Hamiltonian structures of hydrodynamic type (the Poisson brackets of Dubrovin-Novikov type [5]). Corresponding them Hamiltonian structures of hydrodynamic type are strictly nonlocal ([14–16]).

We show the explicit relation between the systems (2.3) and (4.3). For this we shall go over in the equations (2.3) from x, t to new independent variables \tilde{x}, \tilde{t} in according to formulas

$$d\tilde{x} = -\frac{1}{2}(u^1 u^2 + u^1 u^3 + u^2 u^3)dx + u^1 u^2 u^3 dt, \quad d\tilde{t} = dx. \quad (4.4)$$

After the transformation (4.4) the system (2.3) has the form

$$\begin{pmatrix} 1/u^1 \\ 1/u^2 \\ 1/u^3 \end{pmatrix}_{\tilde{t}} = \frac{1}{2} \begin{pmatrix} (u^2 u^3)/u^1 - u^2 - u^3 \\ (u^1 u^3)/u^2 - u^1 - u^3 \\ (u^1 u^2)/u^3 - u^1 - u^2 \end{pmatrix}_{\tilde{x}},$$

i.e., as it is easy to see, it coincides with (4.3) after the transformation

$$w^i = \frac{1}{u^i}. \quad (4.5)$$

Using the language of the initial equations (1.1) and (1.2) we can present the transformations (4.4) and (4.5) in the following way: the equation

$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}$$

goes over into the equation

$$\tilde{f}_{\tilde{x}\tilde{x}\tilde{x}} \tilde{f}_{\tilde{t}\tilde{t}\tilde{t}} - \tilde{f}_{\tilde{x}\tilde{x}\tilde{t}} \tilde{f}_{\tilde{x}\tilde{t}\tilde{t}} = 1$$

after the transformation

$$\begin{aligned} \tilde{x} &= f_{xt}, \quad \tilde{t} = x, \\ \tilde{f}_{\tilde{x}\tilde{x}} &= t, \quad \tilde{f}_{\tilde{x}\tilde{t}} = -f_{xx}, \quad \tilde{f}_{\tilde{t}\tilde{t}} = f_{tt}. \end{aligned} \quad (4.6)$$

Note that this transformation, connecting solutions of the equations associativity (1.1) and (1.2), is not contact.

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